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Approximation of classes of differentiable functions by Poisson integrals

Monograph

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The monograph presents the results of scientific research in the field of approximation theory on classes of differentiable functions. Approximate properties of the Poisson integral on various classes of differentiable functions are considered. For teachers and students of mathematical specialties of higher educational institutions

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Introduction

This monograph investigates the approximations of classes of differentiable functions by λ-methods of summation of their Fourier series and integrals. This methods are determined by the set $\Lambda = \{\lambda_\sigma(\nu)\}$ of continuous for all $\nu \geq 0$ functions $\lambda_\sigma(\nu)$, depending on a real parameter σ from the set $E_\Lambda \subseteq \mathbb{R}$, which has at least one limit point σ_0.

At present, in the field of approximation theory many methods of approximation have been developed by trigonometric polynomials in spaces of periodic functions. Among these methods one can distinguish linear methods that are based on the Fourier sums, and non-linear methods. The linear methods are determined by numerical matrices (Fejer, Vallee Poussin, Zygmund, Rogozinski, Riesz, Korovkin methods, etc.) or by the set of functions (methods of approximation by Poisson integrals and operators).

A. M. Kolmogorov [27] initiated an investigation of approximative properties of linear methods with respect to the classes of differentiation functions. He studied a behaviour of the upper bounds of approximation of functions from the Sobolev classes W^r_∞, $r \in \mathbb{N}$, by their partial Fourier sums $S_n(f;x)$. In particular, he showed that

$$\begin{aligned} \mathcal{E}\left(W^r_\infty; S_n\right)_C = & \sup_{f\in W^r} \|f(x) - S_n(f;x)\|_C = \\ = & \frac{4}{\pi^2}\frac{\ln n}{n^r} + O(n^{-r}), \quad n \to \infty. \end{aligned} \tag{0.1}$$

The research of A. M. Kolmogorov was continued by V. T. Pinkevich [57]. In particular, he found that the ratio (0.1) remains true also for fractional $r > 0$, if under the expression $f^{(r)}(\cdot)$ we understand the derivative in the Weyl sense, i.e., if instead of the classes W^r_∞, the classes $W^r_{r,\infty}$ are considered of functions that are differentiated in the Weyl sense (see definition, [77, p. 126-127]).

Besides the Fourier sums, the other trigonometric polynomials are also considered as approximating aggregates. It is caused by the fact, that in some situations the Fourier sums of a function converges to it very slowly, and even diverge for some continuous 2π-periodic functions. There exists a wide class of methods of construction of approximative linear operators based on the summation of Fourier series.

Denote by $\Lambda = \{\lambda_\delta(k)\}$ the set of functions of natural argument depending on a real parameter δ, which varies on a certain set $E_\Lambda \subseteq R$ with at least one limit point δ_0 and, moreover, $\lambda_\delta(0) = 1$, $\forall \delta \in E_\Lambda$. We note that in the case of $\delta \in \mathbb{N}$ the numbers $\lambda_\delta(k) =: \lambda_{n,k}$ are elements of an infinite rectangular matrix $\Lambda = \{\lambda_{n,k}\}$ $(n, k = 0, 1, ...; \lambda_{n,0} = 1, n \in \mathbb{N} \cup \{0\})$, and with an additional condition $\lambda_{n,k} \equiv 0$, when $k > n$ they are elements of an infinite triangular matrix. Using the set $\{\lambda_\delta(k)\}$ for each function $f \in L_1$ we put into a correspondence the following series

$$\frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \lambda_\delta(k)(a_k(f)\cos(kx) + b_k(f)\sin(kx)), \quad \delta \in E_\Lambda,$$

where $a_0(f)$, $a_k(f)$, $b_k(f)$, $k = 1, ..., n$, are the Fourier coefficients of function f. If this series is for each $\delta \in E_\Lambda$ is the Fourier series for a certain summable function, then we denote it by $U_\delta(f; x; \Lambda)$, and in the case $\delta \in \mathbb{N} \cup \{0\}$ by $U_n(f; x; \Lambda)$. Hence, every set of functions of a natural argument Λ defines a method of construction of polynomials $U_\delta(f; x; \Lambda)$. In other words, it defines a concrete set of polynomial operators $U_\delta\Lambda$ defined on the L_1 set. In this case we say that the set Λ defines a particular method or a process of summation of the Fourier series. It is clear that $\forall \delta \in E_\Lambda$ operators $U_\delta(f; \Lambda)$ are linear. So Λ-methods are also called linear summation methods of the Fourier series.

The next significant step in finding the values of the type (0.1) belongs to S. M. Nikolsky [47]-[49], who summarized listed above results for wider classes of functions and also found an asymptotic equality of the type (0.1) for the case of the Fejer sums for polynomials $U_n(f; x; \Lambda)$ given by a triangular matrix $\Lambda = \{\lambda_{n,k}\}$ with $\lambda_{n,k} = 1 - k/n$,

$n \in \mathbb{N} \cup \{0\},\ k = 0, 1, ..., n-1$. S. M. Nikolsky obtained an asymptotic equality of the type (0.1) in the integral metric. He showed a possibility of spreading the formulation of the Kolmogorov problem as well as its solution into the wider classes of functions, for other approximative aggregates and functional spaces. The results of A. M. Kolmogorov and S. M. Nikolsky initiated a new direction in the theory of approximation of functions and in the theory of Fourier series.

For today, scientific research which originated from the works of A. Lebesgue, Sh. Vallee Poussin, L. Fejer, S. N. Bernstein, A. M. Kolmogorov, B. Nagy, S. M. Nikol'skii, V. K. Dzyadyk, M. P. Korneichuk, I. P. Natanson, S. B. Stechkin, O. V. Efimov, A. I. Stepanets, V. P. Motornyj, S. O. Telyakovskii, O. P. Timan, M. P. Timan and other mathematicians are actively developing in the direction of studying the approximative and extremal properties of linear methods of summation of the Fourier series.

In 1983 A. I. Stepanets (see, [82]-[85], [84, Chapter I], [77, Chapter III]) proposed a new approach to the classification of periodic functions based on the notion of (ψ, β)–derivative. The classes $L^{\psi}_{\beta}\mathfrak{N}$ were introduced, which for a special choice of parameters that determine these classes coincide with the Sobolev classes W^r_∞ and Weyl-Nagy classes $W^r_{\beta,\infty}$. It allowed to classify a wide range of periodic functions and gave a powerful stimulus to new investigations in various directions, in particular, in the study of approximations on the introduced classes by various linear methods of summation of the Fourier series.

For functions given on the real axis, objective functions of exponential type are a natural apparatus of approximation. For the base of the modern theory of approximation of integer functions we refer to the works of M. G. Krein [32], S. N. Bernstein [6], N. I. Akhiezer [1], S. M. Nikolsky [43], O. P. Timan [95, 96], O. P. Timan and M. P. Timan [97], P. Butzer and J. R. Nessel [7] and others. The idea of construction of the theory of approximation of functions given on the real axis, which coveres the theory of approximation of periodic functions belongs to S. N. Bernstein. This idea turned out to be very useful one for both theories. During the last decades they have successfully developed and mutually complement each other in the works of A. I. Stepanets and his followers.

In 1988 A. I. Stepanets [75] introduced classes $\hat{L}^{\psi}_{\beta}\mathfrak{N}$ of locally summable functions, defined on the whole real axis, which in the general case is not periodic and contain, as a partial case, the classes $L^{\psi}_{\beta}\mathfrak{N}$ of periodic functions. For the classes $\hat{L}^{\psi}_{\beta}\mathfrak{N}$ solutions of a number of extreme problems of the approximation theory were obtained using the operators $U_\sigma(\Lambda)$, which are given by kernels having a finite support and with sufficiently general assumptions are integer functions of exponential type, and in the periodic case, λ-methods that are determined by infinite triangular matrices. In particular, the asymptotic formulas were found for the upper bounds of deviations on these classes of Fourier, Fejer, Vallee Poussin, Zygmund, Rogozinski, Steklov, Riesz, Favard, Weierstrass and etc. Concerning the approximation on the classes $\hat{L}^{\psi}_{\beta,1}$ and $\hat{C}^{\psi}_{\beta,\infty}$ by the linear methods that are defined by the set of continuous functions $\lambda_\sigma \nu > 0$, $\nu \in [0, \infty)$, and, in particular, with help of Poisson operators, similar studies were carried out by Yu. I. Kharkevych and T. V. Zhyhallo in [24]-[25], [101].

The topic related to the study of approximative properties of Poisson integrals on different classes of differentiable functions is relevant nowadays and therefore is of a great scientific interest.